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Abstract. We study the homogenization of a diffusion process which takes place in a binary structure formed by an ambiental connected phase surrounding a suspension of very small spheres distributed in an ε-periodic network. The asymptotic distribution of the concentration is determined for both phases, as $\varepsilon \to 0$, assuming that the suspension has mass of unity order and vanishing volume. Three cases are distinguished according to the values of a certain limit capacity. When it is positive and finite, the macroscopic system involves a two-concentration system, coupled through a term accounting for the non local effects. In the other two cases, where the capacity is either infinite or going to zero, although the form of the system is much simpler, some peculiar effects still account for the presence of the suspension.

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1 Introduction

Diffusion occurs naturally and is important in many industrial and geophysical problems, particularly in oil recovery, earth pollution, phase transition, chemical and nuclear processes. When one comes to a rational study of binary structures, a crucial point lies in the interaction between the microscopic and macroscopic levels and particularly the way the former influences the latter. Once the distribution is assumed to be ε-periodic, this kind of study can be accomplished by the homogenization theory.

The present study reveals the basic mechanism which governs diffusion in both phases of such a binary structure, formed by an ambiental connected phase surrounding a periodical suspension of small particles. For simplicity, the particles are considered here to be spheres of radius $r_\varepsilon << \varepsilon$, that is $\lim\limits_{\varepsilon\to 0} \dfrac{r_\varepsilon}{\varepsilon} = 0$. We balance this assumption, which obviously means that the suspension has vanishing volume, by imposing the total mass of the suspension to be always of unity order. This simplified structure permits the accurate establishment of the macroscopic equations by means of a multiple scale method of the homogenization theory adapted for fine-scale substructures. It allows to have a general view on the specific macroscopic effects which arise in every possible case. As we use the non-dimensional framework, the discussion is made in fact with respect to only two parameters: r_ε and b_ε, the latter standing for the ratio of suspension/ambiental phase diffusivities. As the diffusivities of the two components can differ by orders of magnitude, the interfacial conditions play an important role.

It happens that the following cases have different treatments: $r_\varepsilon << \varepsilon^3$, $\varepsilon^3 << r_\varepsilon << \varepsilon$ and $r_\varepsilon = \mathcal{O}(\varepsilon^3)$.

To give a flavor of what may be considered as an appropriate choice of the relative scales, we refer to the pioneering work [7] where the appearance of an extra term in the limit procedure is responsible for a change in the nature of the mathematical problem and is linked to a critical size of the inclusions. Later [6] showed how this could be generalized to the N-dimensional case for non linear operators satisfying classical properties of polynomial growth and coercivity. Since then, the notion of non local effects has been developed in a way that is closer to the present point of view in [4], [5] and [8].

In dealing with our problem, the main difficulty was due to the choice of test functions to be used in the associated variational formulation and which are classically some perturbation of the solution to the so-called cellular problem. Indeed, proceeding as usual in homogenization theory, we use energy arguments based on a priori estimates where direct limiting procedure apparently leads to singular behavior. Non local effects appear when these singularities can be overcome, which is usually achieved by using adequate test functions in the variational formulation. Since the fundamental work [7], an important step was accomplished in this direction in [3]. A slightly different approach [5] uses Dirichlet forms involving non classical measures in the spirit of [10]. However, the main drawback of this method lies in its essential use of the Maximum Principle, which was avoided in [4] for elastic fibers, and later in [8] where the case of spherical symmetry is solved. The asymptotic behavior of highly heterogeneous media has also been considered in the framework of homogenization when the coefficient of one component is vanishing and both components have volumes of unity order: see the derivation of a double porosity model for a single phase flow by [2] and the application of two-scale convergence in order to model diffusion processes in [1].

The paper is organized as follows. Section 2 is devoted to the main notations and to the description of the initial problem. We set the functional framework (16) where the existence and uniqueness of the solution can be established: see [9] and [11] for similar problems. In Section 3, we introduce specific tools to handle the limiting process. This is based on the use of the operators G_r defined by (38) which have a localizing effect: this observation motivates the additional assumption (46) on the external sources when the radius of the particles is of critical order ε^3 with ε denoting the period of the distribution. While passing to the limit, the capacity number γ_ε defined by (33) appears as the main criterium to describe the limit problem, the relative parts played by the radius of the particles and by the period of the network becoming explicit.

Section 4, which is actually the most involving one, deals with the critical case when γ_ε has a positive and finite limit γ. In this part, where we assume also $b_\varepsilon \to +\infty$ the test functions are a convex combination of the elementary solution of the Laplacian and its transformed by the operator G_{r_ε} defined by (38) with $r = r_\varepsilon$. This choice, which is inspired from [3], [4] and [8] and has to be compared with [7], allows to overcome the singular behavior of the energy term when the period ε tends to zero. We have to emphasize that this construction highly depends on the geometry of the problem, that is the spherical symmetry . To our knowledge, the generalization to more intricate geometries remains to be done. The resulting model (69)–(72), with the initial value defined after u_0 in (21) and v_0 in (23), involves a pairing (u, v) which is coupled through a linear operator acting on the difference $u - v$ by the factor $4\pi\gamma$.

The case of the infinite capacity, where $\varepsilon^3 << r_\varepsilon << \varepsilon$, is worked out in Section 5. The proofs are only sketched because the arguments follow the same lines as in Section 4. Let us mention that the singular behavior of the capacity in this case, that is $\gamma_\varepsilon \to +\infty$, forces v to coincide with u. In other words, the infinite capacity prevents the splitting of the distribution, as it did in the critical case. Quite interestingly, the initial value of the global concentration is a convex combination (83) of the initial conditions u_0 and v_0; moreover, the mass density of the macroscopic diffusion equation (82) takes both components into account, in accordance with the intuition that the limiting process must lead to a binary mixture.

Finally, the case of vanishing capacity is handled in Section 6, that is when $r_\varepsilon << \varepsilon^3$. Here, v remains constant in time, obviously equal to the initial condition v_0, while u satisfies the diffusion equation (87)–(88) with data independent of the initial condition of the suspension. This can be seen as a proof that when the radius of the particles is too small, then the suspension does not present macroscopic effects, although a corresponding residual concentration, constant in time, should be considered.

2 The diffusion problem

We consider $\Omega \subseteq \mathbf{R}^3$ a bounded Lipschitz domain occupied by a mixture of two different materials, one of them forming the ambiental connected phase and the other being concentrated in a periodical suspension of small spherical particles. Let us denote

$$Y := \left(-\frac{1}{2}, +\frac{1}{2}\right)^3. \tag{1}$$

$$Y_\varepsilon^k := \varepsilon k + \varepsilon Y, \quad k \in \mathbf{Z}^3. \tag{2}$$

$$\mathbf{Z}_\varepsilon := \{k \in \mathbf{Z}^3, \quad Y_\varepsilon^k \subset \Omega\}, \quad \Omega_{Y_\varepsilon} := \cup_{k \in \mathbf{Z}_\varepsilon} Y_\varepsilon^k. \tag{3}$$

The suspension is defined by the following reunion

$$D_\varepsilon := \cup_{k \in \mathbf{Z}_\varepsilon} B(\varepsilon k, r_\varepsilon), \tag{4}$$

where $0 < r_\varepsilon << \varepsilon$ and $B(\varepsilon k, r_\varepsilon)$ is the ball of radius r_ε centered at εk, $k \in \mathbf{Z}_\varepsilon$. Obviously,

$$|D_\varepsilon| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5}$$

The fluid domain is given by

$$\Omega_\varepsilon = \Omega \setminus D_\varepsilon. \tag{6}$$

We also use the following notation for the cylindrical time-domain:

$$\Omega^T := \Omega \times]0, T[; \tag{7}$$

similar definitions for Ω_ε^T, $\Omega_{Y_\varepsilon}^T$ and D_ε^T.

We consider the problem which governs the diffusion process throughout our binary mixture. Denoting by $a_\varepsilon > 0$ and $b_\varepsilon > 0$ the relative mass density and diffusivity of the suspension, then, assuming without loss of generality that $|\Omega| = 1$, its non-dimensional form is the following:

To find u^ε solution of

$$\rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(k^\varepsilon \nabla u^\varepsilon) = f^\varepsilon \quad \text{in} \quad \Omega^T \tag{8}$$

$$[u^\varepsilon]_\varepsilon = 0 \quad \text{on} \quad \partial D_\varepsilon^T \tag{9}$$

$$[k^\varepsilon \nabla u^\varepsilon]_\varepsilon n = 0 \quad \text{on} \quad \partial D_\varepsilon^T \tag{10}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial \Omega^T \tag{11}$$

$$u^\varepsilon(0) = u_0^\varepsilon \quad \text{in} \quad \Omega \tag{12}$$

where $[\cdot]_\varepsilon$ is the jump across the interface ∂D_ε, n is the normal on ∂D_ε in the outward direction, $f^\varepsilon \in L^2(0, T; H^{-1}(\Omega))$, $u_0^\varepsilon \in L^2(\Omega)$ and

$$\rho^\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega_\varepsilon \\ a_\varepsilon & \text{if} \quad x \in D_\varepsilon \end{cases} \tag{13}$$

$$k^\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in \Omega_\varepsilon \\ b_\varepsilon & \text{if} \quad x \in D_\varepsilon \end{cases} \tag{14}$$

Let H_ε be the Hilbert space $L^2(\Omega)$ endowed with the scalar product

$$(u, v)_{H_\varepsilon} := (\rho^\varepsilon u, v)_\Omega \tag{15}$$

As $H_0^1(\Omega)$ is dense in H_ε for any fixed $\varepsilon > 0$, we can set

$$H_0^1(\Omega) \subseteq H_\varepsilon \simeq H_\varepsilon' \subseteq H^{-1}(\Omega) \tag{16}$$

with continuous embeddings.

Now, we can present the variational formulation of the problem (8)-(12).

To find $u^\varepsilon \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; H_\varepsilon)$ satisfying (in some sense) the initial condition (12) and the following equation

$$\frac{d}{dt}(u^\varepsilon, w)_{H_\varepsilon} + (k_\varepsilon \nabla u^\varepsilon, \nabla w)_\Omega = \langle f^\varepsilon, w \rangle \quad \text{in} \quad \mathcal{D}'(0, T), \quad \forall w \in H_0^1(\Omega) \tag{17}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Theorem 2.1 *Under the above hypotheses and notations, problem (17) has a unique solution. Moreover,* $\dfrac{du^\varepsilon}{dt} \in L^2(0,T;H^{-1}(\Omega))$ *and hence,* u^ε *is equal almost everywhere to a function of* $C^0([0,T];H_\varepsilon)$*; this is the sense of the initial condition (12).*

In the following we consider that the density of the spherical particles is much higher than that of the surrounding phase. The specific feature of our mixture, which describes the fact that although the volume of the suspension is vanishing its mass is of unity order, is given by:

$$\lim_{\varepsilon\to 0} a_\varepsilon |D_\varepsilon| = a > 0 \tag{18}$$

Regarding the relative diffusivity, we only assume:

$$b_\varepsilon \geq b_0 > 0, \quad \forall \varepsilon > 0. \tag{19}$$

As for the data, we assume that there exist $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$ such that

$$f^\varepsilon \rightharpoonup f \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)) \tag{20}$$

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{in} \quad L^2(\Omega) \tag{21}$$

Also, we assume that there exist $C > 0$ (independent of ε) and $v_0 \in L^2(\Omega)$ for which

$$⨍_{D_\varepsilon} |u_0^\varepsilon|^2 dx \leq C \tag{22}$$

$$\frac{1}{|D_\varepsilon|} u_0^\varepsilon \chi_{D_\varepsilon} \rightharpoonup v_0 \quad \text{in} \quad \mathcal{D}'(\Omega) \tag{23}$$

where, for any $D \subset \Omega$, we denote

$$⨍_D \cdot\, dx = \frac{1}{|D|}\int_D \cdot dx.$$

Remark 2.2 *As* u_0^ε *satisfies (22) then (23) holds at least on some subsequence (see Lemma A-2 [3]).*

Proposition 2.3 *We have*

$$u^\varepsilon \quad \textit{is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \tag{24}$$

Moreover, there exists $C > 0$*, independent of* ε*, such that*

$$⨍_{D_\varepsilon} |u^\varepsilon|^2 dx \leq C \quad \textit{a.e. in } [0,T] \tag{25}$$

$$b_\varepsilon |\nabla u^\varepsilon|^2_{L^2(D_\varepsilon^T)} \leq C. \tag{26}$$

Proof. Substituting $w = u^\varepsilon$ in the variational problem (17) and integrating over $(0,t)$ for any $t \in]0,T[$, we get:

$$\frac{1}{2}\left(|u^\varepsilon(t)|^2_{\Omega_\varepsilon} + a_\varepsilon |u^\varepsilon(t)|^2_{D_\varepsilon}\right) + b_\varepsilon \int_0^t |\nabla u^\varepsilon|^2_{D_\varepsilon} ds + \int_0^t |\nabla u^\varepsilon|^2_{\Omega_\varepsilon} ds =$$

$$= \int_0^t \langle f^\varepsilon(s), u^\varepsilon(s)\rangle ds + \frac{1}{2}\left(|u_0^\varepsilon|^2_{\Omega_\varepsilon} + a_\varepsilon |u_0^\varepsilon|^2_{D_\varepsilon}\right).$$

Notice that (21) and (22) yield:

$$|u_0^\varepsilon|^2_{\Omega_\varepsilon} + a_\varepsilon |u_0^\varepsilon|^2_{D_\varepsilon} \leq |u_0^\varepsilon|^2_{\Omega} + a_\varepsilon |D_\varepsilon| \fint_{D_\varepsilon} |u_0^\varepsilon|^2 dx \leq C.$$

Moreover:

$$\int_0^t \langle f^\varepsilon(s), u^\varepsilon(s)\rangle ds \leq \int_0^t |f^\varepsilon|_{H^{-1}} |\nabla u^\varepsilon|_\Omega ds$$

$$\leq \int_0^t |f^\varepsilon|_{H^{-1}} |\nabla u^\varepsilon|_{\Omega_\varepsilon} ds + \int_0^t |f^\varepsilon|_{H^{-1}} |\nabla u^\varepsilon|_{D_\varepsilon} ds$$

$$\leq \frac{1}{2}\int_0^T |f^\varepsilon|^2_{H^{-1}} ds + \frac{1}{2}\int_0^t |\nabla u^\varepsilon|^2_{\Omega_\varepsilon} ds + \frac{1}{2b_\varepsilon}\int_0^T |f^\varepsilon|^2_{H^{-1}} ds + \frac{b_\varepsilon}{2}\int_0^t |\nabla u^\varepsilon|^2_{D_\varepsilon} ds.$$

There results:

$$\frac{1}{2}\left(|u^\varepsilon(t)|^2_{\Omega_\varepsilon} + a_\varepsilon |u^\varepsilon(t)|^2_{D_\varepsilon}\right) + \frac{b_\varepsilon}{2} \int_0^t |\nabla u^\varepsilon|^2_{D_\varepsilon} ds + \frac{1}{2}\int_0^t |\nabla u^\varepsilon|^2_{\Omega_\varepsilon} ds \leq C$$

and the proof is completed. ■

3 Specific tools

First, we introduce

$$\mathcal{R}_\varepsilon = \{R, \quad r_\varepsilon << R << \varepsilon\}$$

that is $R \in \mathcal{R}_\varepsilon$ iff

$$\lim_{\varepsilon \to 0} \frac{r_\varepsilon}{R} = \lim_{\varepsilon \to 0} \frac{R}{\varepsilon} = 0. \tag{27}$$

We have to remark that $\mathcal{R}_\varepsilon$ is an infinite set, this property being insured by the assumption $0 < r_\varepsilon << \varepsilon$.

We denote the domain confined between the spheres of radius a and b by

$$\mathcal{C}(a,b) := \{x \in \mathbf{R}^3,\ a < |x| < b\}$$

and correspondingly

$$\mathcal{C}^k(a,b) := \varepsilon k + \mathcal{C}(a,b).$$

For any $R_\varepsilon \in \mathcal{R}_\varepsilon$, we use the following notations:

$$\mathcal{C}_\varepsilon := \cup_{k \in \mathbf{Z}_\varepsilon} \mathcal{C}^k(r_\varepsilon, R_\varepsilon), \quad \mathcal{C}_\varepsilon^T := \mathcal{C}_\varepsilon \times]0,T[$$

Definition 3.1 *For any $R_\varepsilon \in \mathcal{R}_\varepsilon$, we define $w_{R_\varepsilon} \in H_0^1(\Omega)$ by*

$$w_{R_\varepsilon}(x) \quad := \quad \begin{cases} 0 \quad in \quad \Omega_\varepsilon \setminus \mathcal{C}_\varepsilon, \\ W_{R_\varepsilon}(x - \varepsilon k) \quad in \quad \mathcal{C}_\varepsilon^k, \quad \forall k \in \mathbf{Z}_\varepsilon, \\ 1 \quad in \quad D_\varepsilon. \end{cases} \tag{28}$$

where

$$W_{R_\varepsilon}(y) = \frac{r_\varepsilon}{(R_\varepsilon - r_\varepsilon)}\left(\frac{R_\varepsilon}{|y|} - 1\right) \quad for \quad y \in \mathcal{C}(r_\varepsilon, R_\varepsilon) \tag{29}$$

We have to remark here that $W_{R_\varepsilon} \in H^1(C(r_\varepsilon, R_\varepsilon))$ and satisfies the system

$$\begin{aligned} \Delta W_{R_\varepsilon} &= 0 \quad in \quad \mathcal{C}(r_\varepsilon, R_\varepsilon) && (30)\\ W_{R_\varepsilon} &= 1 \quad for \quad |y| = r_\varepsilon && (31)\\ W_{R_\varepsilon} &= 0 \quad for \quad |y| = R_\varepsilon && (32) \end{aligned}$$

From now on, we denote

$$\gamma_\varepsilon := \frac{r_\varepsilon}{\varepsilon^3}. \tag{33}$$

Proposition 3.2 *For any $R_\varepsilon \in \mathcal{R}_\varepsilon$, we have*

$$|\nabla w_{R_\varepsilon}|_\Omega \leq C\gamma_\varepsilon^{1/2} \tag{34}$$

$$w_{R_\varepsilon} \to 0 \quad in \quad L^2(\Omega). \tag{35}$$

Proof. First notice that

$$|w_{R_\varepsilon}|_\Omega = |w_{R_\varepsilon}|_{\mathcal{C}_\varepsilon \cup D_\varepsilon} \leq |\mathcal{C}_\varepsilon \cup D_\varepsilon|^{1/2} \leq C\left(\frac{R_\varepsilon}{\varepsilon}\right)^{3/2}$$

and $\lim_{\varepsilon \to 0} \frac{R_\varepsilon}{\varepsilon} = 0$ by assumption (27).
As for the rest, direct computation shows

$$\begin{aligned} |\nabla w_{R_\varepsilon}|_\Omega^2 &= \sum_{k \in \mathbf{Z}_\varepsilon} \int_{C^k_{r_\varepsilon, R_\varepsilon}} |\nabla w_{R_\varepsilon}|^2 \, dx \\ &= \sum_{k \in \mathbf{Z}_\varepsilon} \int_0^{2\pi} d\Phi \int_0^\pi \sin\Theta \, d\Theta \int_{r_\varepsilon}^{R_\varepsilon} \frac{dr}{r^2}\left(\frac{r_\varepsilon R_\varepsilon}{R_\varepsilon - r_\varepsilon}\right)^2 \\ &\leq C\frac{|\Omega|}{\varepsilon^3}\left(\frac{1}{r_\varepsilon} - \frac{1}{R_\varepsilon}\right)\left(\frac{r_\varepsilon R_\varepsilon}{R_\varepsilon - r_\varepsilon}\right)^2 \leq C\frac{\gamma_\varepsilon}{(1 - \frac{r_\varepsilon}{R_\varepsilon})} \end{aligned}$$

and the proof is completed by (27). ■

Lemmas 3.3 and 3.4 below are set without proof since they are a three-dimensional adaptation of Lemmas A.3 and A.4 [3].

Lemma 3.3 *For every $0 < r_1 < r_2$ and $u \in H^1(C(r_1, r_2))$, the following estimate holds true:*

$$|\nabla u|^2_{C(r_1,r_2)} \geq \frac{4\pi r_1 r_2}{r_2 - r_1}\left|⨍_{\mathbf{S}_{r_2}} u \, d\sigma - ⨍_{\mathbf{S}_{r_1}} u \, d\sigma\right|^2, \tag{36}$$

where

$$\fint_{\mathbf{S}_r} \cdot \; d\sigma := \frac{1}{4\pi r^2}\int_{\mathbf{S}_r} \cdot \; d\sigma.$$

Lemma 3.4 *There exists a positive constant $C > 0$ such that: $\forall (R,\alpha) \in \mathbf{R}^+ \times (0,1)$, $\forall u \in H^1(B(0,R))$,*

$$\int_{B(0,R)} |u - \fint_{\mathbf{S}_{\alpha R}} u \; d\sigma|^2 \; dx \leq C\frac{R^2}{\alpha}|\nabla u|^2_{B(0,R)}. \tag{37}$$

Definition 3.5 *Consider the piecewise constant functions $G_r : \; L^2(0,T;H^1_0(\Omega)) \to L^2(\Omega^T)$ defined for any $r > 0$ by*

$$G_r(\theta)(x,t) = \sum_{k\in\mathbf{Z}_\varepsilon}\left(\fint_{\mathbf{S}^k_r} \theta(y,t)\; d\sigma_y\right) 1_{Y^k_\varepsilon}(x) \tag{38}$$

where we denote

$$S^k_r = \partial B(\varepsilon k, r). \tag{39}$$

Lemma 3.6 *If $R_\varepsilon \in \mathcal{R}_\varepsilon$, then for every $\theta \in L^2(0,T;H^1_0(\Omega))$ we have*

$$|\theta - G_{R_\varepsilon}(\theta)|_{L^2(\Omega^T_{Y_\varepsilon})} \leq C\left(\frac{\varepsilon^3}{R_\varepsilon}\right)^{1/2}|\nabla\theta|_{L^2(\Omega^T)} \tag{40}$$

$$|\theta - G_{r_\varepsilon}(\theta)|_{L^2(D^T_\varepsilon)} \leq Cr_\varepsilon|\nabla\theta|_{L^2(D^T_\varepsilon)} \tag{41}$$

$$|G_{R_\varepsilon}(\theta) - G_{r_\varepsilon}(\theta)|_{L^2(\Omega^T)} \leq C\left(\frac{\varepsilon^3}{r_\varepsilon}\right)^{1/2}|\nabla\theta|_{L^2(\mathcal{C}^T_\varepsilon)} \tag{42}$$

where $G_{R_\varepsilon}(\theta)$ and $G_{r_\varepsilon}(\theta)$ are defined following (38).

Moreover:

$$|G_{R_\varepsilon}(\theta)|^2_{L^2(\Omega^T)} = \int_0^T\fint_{D_\varepsilon}|G_{R_\varepsilon}(\theta)|^2 dxdt,\; |G_{r_\varepsilon}(\theta)|^2_{L^2(\Omega^T)} = \int_0^T\fint_{D_\varepsilon}|G_{r_\varepsilon}(\theta)|^2 dxdt. \tag{43}$$

Proof. Notice that by definition:

$$\sum_{k\in\mathbf{Z}_\varepsilon}\int_0^T\int_{Y^k_\varepsilon}|\theta - \fint_{\mathbf{S}^k_{R_\varepsilon}}\theta\; d\sigma|^2\; dxdt \leq \sum_{k\in\mathbf{Z}_\varepsilon}\int_0^T\int_{B(\varepsilon k, \frac{\varepsilon\sqrt{3}}{2})}|\theta - \fint_{\mathbf{S}^k_{R_\varepsilon}}\theta\; d\sigma|^2\; dxdt$$

where we have used that

$$Y^k_\varepsilon \subset B(\varepsilon k, \frac{\varepsilon\sqrt{3}}{2})$$

for every $k \in \mathbf{Z}_\varepsilon$. We use Lemma 3.4 with

$$R = \frac{\varepsilon\sqrt{3}}{2}, \quad \alpha = \frac{2R_\varepsilon}{\varepsilon\sqrt{3}}$$

to deduce that

$$\int_{\Omega^T_{Y_\varepsilon}} |\theta - G_{R_\varepsilon}(\theta)|^2\ dxdt \le C\left(\frac{\varepsilon\sqrt{3}}{2}\right)^2 \frac{\varepsilon\sqrt{3}}{2R_\varepsilon}\sum_{k\in\mathbf{Z}_\varepsilon}\int_0^T\int_{B(\varepsilon k,\frac{\varepsilon\sqrt{3}}{2})}|\nabla\theta|^2\ dxdt$$

$$\le C\frac{\varepsilon^3}{R_\varepsilon}\sum_{k\in\mathbf{Z}_\varepsilon}\int_0^T\int_{B(\varepsilon k,\frac{\varepsilon\sqrt{3}}{2})}|\nabla\theta|^2\ dxdt \le C\frac{\varepsilon^3}{R_\varepsilon}\int_{\Omega^T}|\nabla\theta|^2\ dxdt$$

which shows (40).
To establish (41), we recall the definition:

$$\int_{D^T_\varepsilon}|\theta - G_{r_\varepsilon}(\theta)|^2\ dxdt = \sum_{k\in\mathbf{Z}_\varepsilon}\int_0^T\int_{B(\varepsilon k, r_\varepsilon)}|\theta - \fint_{\mathbf{S}^k_{r_\varepsilon}}\theta\ d\sigma|^2\ dxdt$$

Applying Lemma 3.4 with $R = r_\varepsilon$ and $\alpha = 1$, we get the result

$$\int_{D^T_\varepsilon}|\theta - G_{r_\varepsilon}(\theta)|^2\ dxdt\ \le\ Cr_\varepsilon^2\sum_{k\in\mathbf{Z}_\varepsilon}\int_0^T\int_{B(\varepsilon k, r_\varepsilon)}|\nabla\theta|^2\ dxdt \le Cr_\varepsilon^2\int_{D^T_\varepsilon}|\nabla\theta|^2\, dxdt.$$

We come to (42). Indeed, applying Lemma 3.3 and (27):

$$\int_{\Omega^T}|G_{R_\varepsilon}(\theta) - G_{r_\varepsilon}(\theta)|^2\ dxdt = \sum_{k\in\mathbf{Z}_\varepsilon}\int_0^T\int_{Y^k_\varepsilon}|\fint_{\mathbf{S}^k_{R_\varepsilon}}\theta\ d\sigma - \fint_{\mathbf{S}^k_{r_\varepsilon}}\theta\ d\sigma|^2\ dydt$$

$$\le \sum_{k\in\mathbf{Z}_\varepsilon}\int_{Y^k_\varepsilon}\frac{(R_\varepsilon - r_\varepsilon)}{4\pi R_\varepsilon r_\varepsilon}\ dy\int_0^T\int_{C^k_{r_\varepsilon,R_\varepsilon}}|\nabla\theta|^2\ dxdt = \frac{(R_\varepsilon - r_\varepsilon)}{4\pi r_\varepsilon R_\varepsilon}\sum_{k\in\mathbf{Z}_\varepsilon}\varepsilon^3\int_0^T\int_{C^k_{r_\varepsilon,R_\varepsilon}}|\nabla\theta|^2\, dxdt$$

$$= C\varepsilon^3\frac{(R_\varepsilon - r_\varepsilon)}{4\pi r_\varepsilon R_\varepsilon}\int_{\mathcal{C}^T_\varepsilon}|\nabla\theta|^2\ dxdt \le C\frac{\varepsilon^3}{r_\varepsilon}\int_{\mathcal{C}^T_\varepsilon}|\nabla\theta|^2\ dxdt.$$

Finally, a direct computation yields (43). ■

Proposition 3.7 *If $R_\varepsilon \in \mathcal{R}_\varepsilon$, then for any $\theta \in L^2(0,T;H^1_0(\Omega))$ there holds true:*

$$\int_0^T\fint_{D_\varepsilon}|\theta|^2\ dxdt \le C\max(1,\frac{\varepsilon^3}{r_\varepsilon})|\nabla\theta|^2_{L^2(\Omega^T)}.$$

Proof. We have:

$$\int_0^T\fint_{D_\varepsilon}|\theta|^2\ dxdt \le 2\int_0^T\fint_{D_\varepsilon}|\theta - G_{r_\varepsilon}(\theta)|^2\ dxdt + 2\int_0^T\fint_{D_\varepsilon}|G_{r_\varepsilon}(\theta)|^2\ dxdt$$

$$= 2\int_0^T\fint_{D_\varepsilon}|\theta - G_{r_\varepsilon}(\theta)|^2\ dxdt + 2\int_{\Omega^T}|G_{r_\varepsilon}(\theta)|^2\ dxdt$$

$$\le Cr_\varepsilon^2\int_0^T\fint_{D_\varepsilon}|\nabla\theta|^2\ dxdt + 4\int_{\Omega^T}|G_{r_\varepsilon}(\theta) - G_{R_\varepsilon}(\theta)|^2\ dxdt +$$

$$+8\int_{\Omega^T}|G_{R_\varepsilon}(\theta) - \theta|^2\ dxdt + 8\int_{\Omega^T}|\theta|^2\ dxdt$$

$$\leq C r_\varepsilon^2 \int_0^T \fint_{D_\varepsilon} |\nabla\theta|^2 \, dxdt + C\frac{\varepsilon^3}{r_\varepsilon} \int_{\mathcal{C}_\varepsilon^T} |\nabla\theta|^2 \, dxdt +$$

$$+ C\frac{\varepsilon^3}{R_\varepsilon} \int_{\Omega^T} |\nabla\theta|^2 \, dxdt + C \int_{\Omega^T} |\nabla\theta|^2 \, dxdt$$

$$\leq C\left(\frac{\varepsilon^3}{r_\varepsilon} + \frac{\varepsilon^3}{R_\varepsilon} + 1\right) \int_{\Omega^T} |\nabla\theta|^2 \, dxdt \leq C \max(1, \frac{\varepsilon^3}{r_\varepsilon}) \int_{\Omega^T} |\nabla\theta|^2 \, dxdt$$

■

Remark 3.8 *Using the Mean Value Theorem, we easily find that*

$$|G_{r_\varepsilon}(\varphi) - \varphi|_{L^\infty(\mathcal{C}_\varepsilon \cup D_\varepsilon)} \leq 2R_\varepsilon |\nabla\varphi|_{L^\infty(\Omega)}, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Definition 3.9 *Let* $M_{D_\varepsilon} : L^2(0,T; C_c(\Omega)) \to L^2(\Omega^T)$ *be defined by*

$$M_{D_\varepsilon}(\varphi)(x,t) := \sum_{k \in \mathbf{Z}_\varepsilon} \left(\fint_{Y_\varepsilon^k} \varphi(y,t) \, dy \right) 1_{B(\varepsilon k, r_\varepsilon)}(x).$$

Lemma 3.10 *For any* $\varphi \in L^2(0,T; C_c(\Omega))$*, we have:*

$$\lim_{\varepsilon \to 0} \int_0^T \fint_{D_\varepsilon} |\varphi - M_{D_\varepsilon}(\varphi)|^2 dxdt = 0.$$

Proof. Notice that

$$\fint_{D_\varepsilon} |\varphi - M_{D_\varepsilon}(\varphi)|^2 dx = \frac{1}{|D_\varepsilon|} \sum_{k \in \mathbf{Z}_\varepsilon} \int_{B(\varepsilon k, r_\varepsilon)} |\varphi - \fint_{Y_\varepsilon^k} \varphi \, dy|^2 \, dx.$$

As $\text{card}(\mathbf{Z}_\varepsilon) \simeq \frac{|\Omega|}{\varepsilon^3}$, then $|B(0,r_\varepsilon)| \frac{\text{card}(\mathbf{Z}_\varepsilon)}{|D_\varepsilon|} \to |\Omega| = 1$ and by the uniform continuity of φ on Ω it follows the convergence to 0 a.e. on $[0,T]$. Lebesgue's dominated convergence theorem achieves the result. ■

4 Homogenization of the case $r_\varepsilon = \mathcal{O}(\varepsilon^3)$

The present critical radius case is described by

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon = \gamma \in]0, +\infty[. \tag{44}$$

Its homogenization process is the most involving one. That is why we start the homogenization study of our problem with this case, under the condition

$$\lim_{\varepsilon \to 0} b_\varepsilon = +\infty \tag{45}$$

We also assume that f^ε has the following additional property:

$$\begin{cases} \exists R_\varepsilon \in \mathcal{R}_\varepsilon \quad \text{and} \quad g \in L^2(0,T; H^{-1}(\Omega)) \quad \text{for which} \\ \\ \langle f^\varepsilon, w_{R_\varepsilon}\varphi \rangle \to \langle g, \varphi \rangle \quad \text{in} \quad \mathcal{D}'(0,T), \quad \forall \varphi \in \mathcal{D}(\Omega) \end{cases} \tag{46}$$

(see [8] for a certain type of functions f^ε which satisfy (46)).

Remark 4.1 *Notice that due to (44), in this case Proposition 3.7 reads*

$$\forall \varphi \in L^2(0,T;H_0^1(\Omega)), \qquad \int_0^T \fint_{D_\varepsilon} |\varphi|^2 \, dxdt \leq C|\nabla\varphi|^2_{L^2(\Omega^T)}. \tag{47}$$

A preliminary result is the following:

Proposition 4.2 *There exist $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and $v \in L^2(\Omega^T)$ such that, on some subsequence,*

$$u^\varepsilon \overset{\star}{\rightharpoonup} u \quad in \quad L^\infty(0,T;L^2(\Omega)) \tag{48}$$

$$u^\varepsilon \rightharpoonup u \quad in \quad L^2(0,T;H_0^1(\Omega)) \tag{49}$$

$$G_{R_\varepsilon}(u^\varepsilon) \to u \quad in \quad L^2(\Omega^T) \tag{50}$$

$$G_{r_\varepsilon}(u^\varepsilon) \rightharpoonup v \quad in \quad L^2(\Omega^T) \tag{51}$$

Moreover, we have

$$\lim_{\varepsilon\to 0} \int_0^T \fint_{D_\varepsilon} |u^\varepsilon - G_{r_\varepsilon}(u^\varepsilon)|^2 dxdt = 0 \tag{52}$$

Proof. From (24), we get, on some subsequence, the convergences (48) and (49). Moreover, we have:

$$|u - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T} = |u|^2_{\Omega^T \setminus \Omega^T_{Y_\varepsilon}} + |u - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T_{Y_\varepsilon}} \tag{53}$$

where:

$$\begin{aligned} |u - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} &\leq |u - u^\varepsilon|_{\Omega^T_{Y_\varepsilon}} + |u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} \\ &\leq |u - u^\varepsilon|_{\Omega^T} + |u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} \end{aligned} \tag{54}$$

and (40) yields:

$$|u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T_{Y_\varepsilon}} \leq C\frac{\varepsilon^3}{R_\varepsilon}|\nabla u^\varepsilon|^2_{\Omega^T} = C\frac{\varepsilon^3}{r_\varepsilon}\frac{r_\varepsilon}{R_\varepsilon}|\nabla u^\varepsilon|^2_{\Omega^T} \leq C\frac{r_\varepsilon}{R_\varepsilon}$$

and thus:

$$\lim_{\varepsilon\to 0} |u^\varepsilon - G_{R_\varepsilon}(u^\varepsilon)|^2_{\Omega^T_{Y_\varepsilon}} = 0.$$

As (49) implies that

$$u^\varepsilon \to u \quad \text{in} \quad L^2(\Omega^T) \tag{55}$$

the right-hand side of (54) tends to zero as $\varepsilon \to 0$, that is:

$$\lim_{\varepsilon\to 0} |u - G_{R_\varepsilon}(u^\varepsilon)|_{\Omega^T_{Y_\varepsilon}} = 0.$$

After substitution into the right-hand side of (53), and taking into account that

$$\lim_{\varepsilon\to 0} |\Omega^T \setminus \Omega^T_{Y_\varepsilon}| = 0,$$

we obtain (50), that is,

$$G_{R_\varepsilon}(u^\varepsilon) \to u \quad \text{in} \quad L^2(\Omega^T). \tag{56}$$

In order to prove (51), we see that

$$\begin{aligned} & |G_{r_\varepsilon}(u^\varepsilon)|_{L^2(\Omega^T)} \leq |G_{r_\varepsilon}(u^\varepsilon) - G_{R_\varepsilon}(u^\varepsilon)|_{L^2(\Omega^T)} + |G_{R_\varepsilon}(u^\varepsilon)|_{L^2(\Omega^T)} \\ & \leq \frac{C}{\gamma_\varepsilon^{1/2}} |\nabla u^\varepsilon|_{L^2(\Omega^T)} + C \leq C. \end{aligned} \tag{57}$$

Moreover, recall that from (41) we have, taking into account (26):

$$\int_0^T \fint_{D_\varepsilon} |u^\varepsilon - G_{r_\varepsilon}(u^\varepsilon)|^2 \, dxdt \leq C r_\varepsilon^2 \int_0^T \fint_{D_\varepsilon} |\nabla u^\varepsilon|^2 \, dxdt \leq \frac{C}{\gamma_\varepsilon b_\varepsilon} \to 0 \tag{58}$$

and the proof is completed. ∎

Proposition 4.3 *For any $\varphi \in L^2(0,T;C_c(\Omega))$, we have:*

$$\lim_{\varepsilon \to 0} \int_0^T \fint_{D_\varepsilon} u^\varepsilon \varphi dxdt = \int_{\Omega^T} v\varphi dxdt. \tag{59}$$

Proof. We have:

$$\begin{aligned} & \int_0^T \fint_{D_\varepsilon} u^\varepsilon \varphi dxdt = \int_0^T \fint_{D_\varepsilon} (u^\varepsilon - G_{r_\varepsilon}(u^\varepsilon))\varphi dxdt + \\ & + \int_0^T \fint_{D_\varepsilon} G_{r_\varepsilon}(u^\varepsilon)(\varphi - M_{D_\varepsilon}(\varphi)) dxdt + \int_0^T \fint_{D_\varepsilon} G_{r_\varepsilon}(u^\varepsilon) M_{D_\varepsilon}(\varphi) dxdt \end{aligned} \tag{60}$$

The first right-hand term tends to zero thanks to (52) in Proposition 4.2. The second one tends also to zero thanks to Lemma 3.10. The last term is handled as follows:

$$\int_0^T \fint_{D_\varepsilon} G_{r_\varepsilon}(u^\varepsilon) M_{D_\varepsilon}(\varphi) dxdt = \lambda_\varepsilon \sum_{k \in \mathbf{Z}_\varepsilon} \int_0^T \int_{Y_\varepsilon^k} \left(\fint_{\mathbf{S}_\varepsilon^k} u^\varepsilon d\sigma \right) \varphi dxdt = \lambda_\varepsilon \int_{\Omega^T} \varphi G_{r_\varepsilon}(u^\varepsilon) dxdt$$

where

$$\lambda_\varepsilon := \frac{|B(0,r_\varepsilon)|}{\varepsilon^3 |D_\varepsilon|} \to 1 \quad \text{as} \quad |\Omega| = 1.$$

The proof is completed by (51). ∎

Proposition 4.4 *For any $\varphi \in L^2(0,T;H_0^1(\Omega))$, we have*

$$\int_0^T \fint_{D_\varepsilon} u^\varepsilon \varphi dxdt \to \int_{\Omega^T} v\varphi dxdt. \tag{61}$$

Proof. In the light of proposition 4.3, we have to prove that the left-hand side term is continuous in the corresponding norm. This can be obtained as follows:

$$\left| \int_0^T \fint_{D_\varepsilon} u^\varepsilon \varphi dxdt \right| \leq \left(\int_0^T \fint_{D_\varepsilon} |u^\varepsilon|^2 dxdt \right)^{1/2} \left(\int_0^T \fint_{D_\varepsilon} |\varphi|^2 dxdt \right)^{1/2} \leq$$

$$\leq C|\varphi|^2_{L^2(0,T;H^1_0(\Omega))},$$

where we used (25) and (47). ■

Proposition 4.5 *Let for any $R_\varepsilon \in \mathcal{R}_\varepsilon$ and $\varphi, \psi \in \mathcal{D}(\Omega)$*

$$\Phi^\varepsilon = (1 - w_{R_\varepsilon})\varphi + w_{R_\varepsilon} G_{r_\varepsilon}(\psi) \tag{62}$$

Then, for any $\eta \in \mathcal{D}([0,T[)$, we have

$$\lim_{\varepsilon\to 0} |\Phi^\varepsilon - \varphi|_\Omega = 0 \tag{63}$$

$$\lim_{\varepsilon\to 0} \int_{\Omega^T} \rho^\varepsilon u^\varepsilon \Phi^\varepsilon \eta'(t) dxdt = \int_{\Omega^T} u\varphi\eta'(t) dxdt + a\int_{\Omega^T} v\psi\eta'(t) dxdt. \tag{64}$$

Proof. The property (63) is an immediate consequence of (35) and of the uniform boundness of $G_{r_\varepsilon}(\psi)$ in $L^\infty(\Omega)$.
For the second property, let us notice that

$$\int_{\Omega^T} \rho^\varepsilon u^\varepsilon \Phi^\varepsilon \eta'(t) dxdt = \int_0^T\!\!\int_\Omega \chi_{\Omega_\varepsilon} u^\varepsilon \Phi^\varepsilon(x)\eta'(t) dxdt$$
$$+ a_\varepsilon \int_0^T \int_{D_\varepsilon} u^\varepsilon G_{r_\varepsilon}(\psi)\eta'(t) dxdt.$$

As we obviously have

$$\lim_{\varepsilon\to 0} \int_0^T\!\!\int_\Omega \chi_{\Omega_\varepsilon} u^\varepsilon \Phi^\varepsilon(x)\eta'(t) dxdt = \int_{\Omega^T} u\varphi\eta'(t) dxdt,$$

it remains to study

$$a_\varepsilon \int_0^T \int_{D_\varepsilon} u^\varepsilon G_{r_\varepsilon}(\psi)\eta'(t) dxdt = a_\varepsilon |D_\varepsilon| \int_0^T \fint_{D_\varepsilon} u^\varepsilon G_{r_\varepsilon}(\psi)\eta'(t) dxdt.$$

Using (59) and the uniform continuity of ψ, we get

$$\lim_{\varepsilon\to 0} a_\varepsilon \int_0^T \int_{D_\varepsilon} u^\varepsilon G_{r_\varepsilon}(\psi)\eta'(t) dxdt = a\int_{\Omega^T} v\psi\eta' dxdt.$$

■

Proposition 4.6 *If Φ^ε is defined like in Proposition 4.5, then we have*

$$\lim_{\varepsilon\to 0} \int_0^T\!\!\int_\Omega \nabla u^\varepsilon \cdot \nabla\Phi^\varepsilon \eta(t)\ dxdt = \int_{\Omega^T} \nabla u \cdot \nabla\varphi\eta(t)\ dxdt + 4\pi\gamma \int_{\Omega^T} (v-u)(\psi-\varphi)\eta(t)\ dxdt \tag{65}$$

Proof. First consider

$$\int_0^T\!\!\int_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla\Phi^\varepsilon\ dxdt$$

which reduces to

$$\int_0^T\int_{\Omega_\varepsilon\setminus\mathcal{C}_\varepsilon} \nabla u^\varepsilon\cdot\nabla\varphi\eta\, dxdt + \int_0^T\int_{\mathcal{C}_\varepsilon} \nabla u^\varepsilon\cdot\nabla\Phi^\varepsilon\eta dxdt.$$

Lebesgue's dominated convergence theorem yields $\nabla\varphi 1_{\Omega_\varepsilon\setminus\mathcal{C}_\varepsilon}\to\nabla\varphi$ in $L^2(\Omega)$. Thus, taking (49) into account

$$\int_0^T\int_\Omega \nabla u^\varepsilon\cdot\nabla\varphi\,\eta\chi_{\Omega_\varepsilon\setminus\mathcal{C}_\varepsilon}\, dxdt \to \int_{\Omega^T}\nabla u\cdot\nabla\varphi\,\eta\, dxdt.$$

Now, we come to the remaining part, namely

$$\begin{aligned}\int_0^T\int_{\mathcal{C}\varepsilon}\nabla u^\varepsilon\cdot\nabla\Phi^\varepsilon\eta(t)\, dxdt &= \int_0^T\int_{\mathcal{C}\varepsilon}(1-w_{R_\varepsilon})\nabla u^\varepsilon\cdot\nabla\varphi\eta\, dxdt\\ +\int_0^T\int_{\mathcal{C}\varepsilon}\nabla u^\varepsilon\cdot\nabla w_{R_\varepsilon}(G_{r_\varepsilon}(\psi)-\varphi)dxdt&\\ := \quad I_1+I_2&\end{aligned}\tag{66}$$

In the first integral, as $\chi_{\mathcal{C}_\varepsilon}\nabla\varphi\to 0$ in $L^2(\Omega^T)$, $\nabla u^\varepsilon\rightharpoonup\nabla u$ in $L^2(\Omega^T)$ and $(1-w_{R_\varepsilon})$ is obviously bounded, we easily find that I_1 tends to zero.
In order to study I_2, let us notice that

$$\begin{aligned}I_2 \quad &= \int_0^T\int_{\mathcal{C}\varepsilon}\nabla u^\varepsilon\cdot\nabla w_{R_\varepsilon}(G_{r_\varepsilon}(\varphi)-\varphi)\,\eta\, dxdt+\\ &\quad +\int_0^T\int_{\mathcal{C}\varepsilon}\nabla u^\varepsilon\cdot\nabla w_{R_\varepsilon}(G_{r_\varepsilon}(\psi)-G_{r_\varepsilon}(\varphi))\,\eta\, dxdt\end{aligned}\tag{67}$$

The first term in the right-hand side of (67) may be estimated by

$$|\int_0^T\int_{\mathcal{C}\varepsilon}\nabla u^\varepsilon\cdot\nabla w_{R_\varepsilon}(\varphi-G_{r_\varepsilon}(\varphi))\,\eta\, dxdt|\le|\nabla u^\varepsilon|_{\Omega^T}|\nabla w_{R_\varepsilon}\eta|_{\Omega^T}|\varphi-G_{r_\varepsilon}(\varphi)|_{L^\infty(\mathcal{C}_\varepsilon)}.\tag{68}$$

As (w_{R_ε}) is bounded in $H^1(\Omega)$ (*see* Proposition 3.2), the right hand side of (68) tends to zero by Remark 3.8.
Going back to the second term in the right hand side of (67), we may write

$$\begin{aligned}&\int_0^T\int_{\mathcal{C}\varepsilon}\nabla u^\varepsilon\cdot\nabla w_{R_\varepsilon}(G_{r_\varepsilon}(\psi)-G_{r_\varepsilon}(\varphi))\eta(t)\, dxdt\\ &= \sum_{k\in\mathbf{Z}_\varepsilon}\left(\fint_{\mathbf{S}^k_{r_\varepsilon}}\psi\, d\sigma-\fint_{\mathbf{S}^k_{r_\varepsilon}}\varphi\, d\sigma\right)\int_0^{2\pi}d\Phi\int_0^\pi\sin\Theta\, d\Theta\int_{r_\varepsilon}^{R_\varepsilon}\left(\int_0^T\frac{\partial u^\varepsilon}{\partial r}\bigg|_{\mathcal{C}^k(r_\varepsilon,R_\varepsilon)}\eta(t)dt\right)\frac{dW_{R_\varepsilon}}{dr}r^2\, dr\\ &= \frac{r_\varepsilon R_\varepsilon}{(R_\varepsilon-r_\varepsilon)}\sum_{k\in\mathbf{Z}_\varepsilon}\left(\fint_{\mathbf{S}^k_{r_\varepsilon}}\psi\, d\sigma-\fint_{\mathbf{S}^k_{r_\varepsilon}}\varphi\, d\sigma\right)\int_{\mathbf{S}_1}\int_0^T(u^\varepsilon|_{|x-\varepsilon k|=r_\varepsilon}-u^\varepsilon|_{|x-\varepsilon k|=R_\varepsilon})\eta(t)dtd\sigma_1\\ &= \frac{4\pi r_\varepsilon R_\varepsilon}{\varepsilon^3(R_\varepsilon-r_\varepsilon)}\int_{\Omega^T}(G_{r_\varepsilon}(u^\varepsilon)-G_{R_\varepsilon}(u^\varepsilon))(G_{r_\varepsilon}(\psi)-G_{r_\varepsilon}(\varphi))\eta(t)\, dxdt\end{aligned}$$

from which we infer that I_2 is converging to

$$4\pi\gamma\int_{\Omega^T}(v-u)(\psi-\varphi)\eta(t)\, dxdt$$

and the proof is completed. ■

We are in the position to state our main result:

Theorem 4.7 *The limits* $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ *and* $v \in L^2(\Omega^T)$ *of (48)–(51) verify (in a weak sense) the following problem:*

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u + 4\pi\gamma(u-v) &= (f-g) \quad in \quad \Omega^T, & (69)\\ a\frac{\partial v}{\partial t} + 4\pi\gamma(v-u) &= g \quad in \quad \Omega^T, & (70)\\ u(0) &= u_0 \quad in \quad \Omega & (71)\\ v(0) &= v_0 \quad in \quad \Omega & (72)\end{aligned}$$

Moreover, there holds $u \in C^0([0,T];L^2(\Omega))$ *and* $v \in C^0([0,T];H^{-1}(\Omega))$*; these are the senses of (71) and (72).*

Remark 4.8 *As the problem (69)–(72) has a unique weak solution, the convergences in Proposition 4.2 hold on the whole sequence.*

Proof of Theorem 4.7. We set in (17) $w = \Phi^\varepsilon$ where Φ^ε is defined like in lemma 4.5. Then, by multiplying (17) by $\eta \in \mathcal{D}([0,T[)$ and integrating it over $[0,T]$ we get

$$-\int_{\Omega^T} \rho^\varepsilon u^\varepsilon \Phi^\varepsilon \eta' dxdt + \int_{\Omega^T} k^\varepsilon \nabla u^\varepsilon (\nabla \Phi^\varepsilon)\eta dxdt = \int_0^T \langle f^\varepsilon, \Phi^\varepsilon\rangle \eta dt + \int_\Omega \rho^\varepsilon u_0^\varepsilon \Phi^\varepsilon \eta(0)dx. \quad (73)$$

Then, the left-hand side tends to

$$\begin{aligned} -\int_{\Omega^T} u\varphi\eta' dxdt - a\int_{\Omega^T} v\varphi\eta' dxdt + \int_{\Omega^T} \nabla u \cdot \nabla\varphi\, \eta\; dxdt + \\ +4\pi\gamma \int_{\Omega^T} (v-u)(\psi-\varphi)\,\eta\; dxdt. \end{aligned} \quad (74)$$

This is a direct consequence of Proposition 4.6 together with the remark that

$$\int_0^T \int_{D_\varepsilon} \nabla u^\varepsilon \nabla \Phi^\varepsilon \; dxdt = 0$$

since Φ^ε is constant on every $B(\varepsilon k, r_\varepsilon)$, $k \in \mathbf{Z}_\varepsilon$.
As for the right-hand side, we have

$$\int_0^T \langle f^\varepsilon, \Phi^\varepsilon\rangle \eta dt = \int_0^T \langle f^\varepsilon, (1-w_{R_\varepsilon})\varphi\rangle \eta dt + \int_0^T \langle f^\varepsilon, w_{R_\varepsilon} G_{r_\varepsilon}(\psi)\rangle \eta dt$$

and, with hypothesis (46),

$$\int_0^T \langle f^\varepsilon, (1-w_{R_\varepsilon})\varphi\rangle \eta dt \to \int_0^T \langle f-g, \varphi\rangle \eta dt.$$

Moreover,

$$\int_0^T \langle f^\varepsilon, w_{R_\varepsilon} G_{r_\varepsilon}(\psi)\rangle \eta dt = \int_0^T \langle f^\varepsilon, w_{R_\varepsilon}(G_{r_\varepsilon}(\psi)-\psi)\rangle \eta dt + \int_0^T \langle f^\varepsilon, w_{R_\varepsilon}\psi\rangle \eta dt$$

with

$$\left| \int_0^T \langle f^\varepsilon, w_{R_\varepsilon}(G_{r_\varepsilon}(\psi) - \psi) \rangle \eta dt \right| \leq \int_0^T |f^\varepsilon|_{H^{-1}} |w_{R_\varepsilon}(G_{r_\varepsilon}(\psi) - \psi)|_{H_0^1(\Omega)}.$$

As we have

$$\begin{aligned} |w_{R_\varepsilon}(G_{r_\varepsilon}(\psi) - \psi)|_{H_0^1(\Omega)} &= |\nabla(w_{R_\varepsilon}(G_{r_\varepsilon}(\psi) - \psi))|_\Omega \\ &\leq |\nabla w_{R_\varepsilon}|_{\mathcal{C}_\varepsilon} |G_{r_\varepsilon}(\psi) - \psi|_{L^\infty(\mathcal{C}_\varepsilon)} + |\nabla \psi|_{\mathcal{C}_\varepsilon \cup D_\varepsilon} \end{aligned}$$

Remark 3.8 and (34) obviously yield

$$\lim_{\varepsilon \to 0} |w_{R_\varepsilon}(G_{r_\varepsilon}(\psi) - \psi)|_{H_0^1(\Omega)} = 0.$$

The assumption (20) on f^ε implies that $|f^\varepsilon|_{H^{-1}} \leq C$ and thus

$$\lim_{\varepsilon \to 0} \int_0^T \langle f^\varepsilon, w_{R_\varepsilon}(G_{r_\varepsilon}(\psi) - \psi) \rangle \eta dt = 0.$$

We conclude thanks to hypothesis (46) that

$$\lim_{\varepsilon \to 0} \int_0^T \langle f^\varepsilon, w_{R_\varepsilon} \psi \rangle \eta dt = \int_0^T \langle g,\, \psi \rangle \eta dt.$$

Finally:

$$\lim_{\varepsilon \to 0} \int_0^T \langle f^\varepsilon, \Phi^\varepsilon \rangle \eta dt = \int_0^T \langle f - g,\, \varphi \rangle \eta dt + \int_0^T \langle g,\, \psi \rangle \eta dt.$$

We get

$$\int_\Omega \rho^\varepsilon u_0^\varepsilon \Phi^\varepsilon \eta(0) dx = \int_{\Omega_\varepsilon} u_0^\varepsilon \Phi^\varepsilon \eta(0) dx + a_\varepsilon \int_{D_\varepsilon} u_0^\varepsilon G_{r_\varepsilon}(\psi) \eta(0) dx.$$

Using the hypotheses (21)–(23) on u_0^ε, we pass to the limit and with the same arguments as above we obtain

$$\lim_{\varepsilon \to 0} \int_\Omega \rho^\varepsilon u_0^\varepsilon \Phi^\varepsilon \eta(0) dx = \eta(0) \int_\Omega (u_0 \varphi + a v_0 \psi) dx$$

which achieves the proof. ∎

5 Homogenization in the case $\varepsilon^3 << r_\varepsilon << \varepsilon$

In this section, we fix some $R_\varepsilon \in \mathcal{R}_\varepsilon$.

Remark 5.1 *Notice that in this case Proposition 3.7 also reads*

$$\int_0^T \fint_{D_\varepsilon} |\varphi|^2 \, dx dt \leq C |\nabla \varphi|^2_{L^2(\Omega^T)}, \quad \forall \varphi \in L^2(0, T; H_0^1(\Omega)). \tag{75}$$

In the present case, Proposition 2.3 and Lemma 3.6 imply in a straightforward manner the result corresponding to Proposition 4.2.

Proposition 5.2 *There exists $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ such that, on some subsequence,*

$$u^\varepsilon \overset{\star}{\rightharpoonup} u \quad in \quad L^\infty(0,T;L^2(\Omega)) \tag{76}$$

$$u^\varepsilon \rightharpoonup u \quad in \quad L^2(0,T;H_0^1(\Omega)) \tag{77}$$

$$G_{R_\varepsilon}(u^\varepsilon) \to u \quad in \quad L^2(\Omega^T) \tag{78}$$

$$G_{r_\varepsilon}(u^\varepsilon) \to u \quad in \quad L^2(\Omega^T) \tag{79}$$

Moreover, we have

$$\lim_{\varepsilon\to 0} \int_0^T \fint_{D_\varepsilon} |u^\varepsilon - G_{r_\varepsilon}(u^\varepsilon)|^2 dxdt = 0 \tag{80}$$

In the light of Remark 5.1, we prove as in the previous section:

Proposition 5.3 *For any $\varphi \in L^2(0,T;H_0^1(\Omega))$, we have*

$$\int_0^T \fint_{D_\varepsilon} u^\varepsilon \varphi dxdt \to \int_{\Omega^T} u\varphi dxdt. \tag{81}$$

The homogenization result obtained in this case follows.

Theorem 5.4 *The limit $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ of (76)–(79) is the only solution of*

$$(1+a)\frac{\partial u}{\partial t} - \Delta u = f \quad in \quad \Omega^T, \tag{82}$$

$$u(0) = \frac{1}{(1+a)}u_0 + \frac{a}{(1+a)}v_0 \quad in \quad \Omega \tag{83}$$

Moreover, the convergences in Proposition 5.2 hold on the whole sequence and $u \in C^0([0,T];L^2(\Omega))$, this being the sense of (83).

Proof. The proof of (82) is similar to the corresponding one of the Theorem 4.7. The test function Φ^ε is given by

$$\Phi^\varepsilon = (1 - w_{R_\varepsilon})\varphi + w_{R_\varepsilon} G_{r_\varepsilon}(\varphi), \quad \varphi \in \mathcal{D}(\Omega).$$

The only interesting convergences are the following two:

$$\left| \int_{\mathcal{C}_\varepsilon^T} \nabla u^\varepsilon (\nabla w_{R_\varepsilon})(G_{r_\varepsilon}(\varphi) - \varphi) dxdt \right| \le C|\nabla u^\varepsilon|_{\Omega^T} |\nabla w_{R_\varepsilon}|_{\Omega^T} \, |G_{r_\varepsilon}(\varphi) - \varphi|_{L^\infty(\mathcal{C}_\varepsilon^T)} \le$$

$$\le C\gamma_\varepsilon^{1/2} R_\varepsilon = C\left(\frac{r_\varepsilon}{\varepsilon}\right)^{1/2} \left(\frac{R_\varepsilon}{\varepsilon}\right) \to 0$$

$$\left| \int_0^T \langle f^\varepsilon, w_{R_\varepsilon}(G_{r_\varepsilon}(\varphi) - \varphi) \rangle \right| \le C\, |(G_{r_\varepsilon}(\varphi) - \varphi)\nabla w_{R_\varepsilon}|_{L^2(\mathcal{C}_\varepsilon^T)} + C\, |w_{R_\varepsilon} \nabla\varphi|_{L^2(\mathcal{C}_\varepsilon^T \cup D_\varepsilon^T)} \le$$

$$\leq C\,|\nabla\varphi|_{L^\infty(\Omega)}\left(\gamma_\varepsilon^{1/2}R_\varepsilon+|\mathcal{C}_\varepsilon\cup D_\varepsilon|^{1/2}\right)\to 0,$$

where we have used the a priori estimates of Proposition 2.3, Remark 3.8 and Proposition 3.2.
Using Proposition 5.2 and hypotheses (21)–(23), we obtain with the same argument as before

$$\lim_{\varepsilon\to 0}\int_\Omega \rho^\varepsilon u_0^\varepsilon\Phi^\varepsilon\eta(0)dx=\eta(0)\int_\Omega(u_0+av_0)\varphi dx$$

which achieves the proof. ■

6 Homogenization in the case $r_\varepsilon << \varepsilon^3$.

As in this case $\gamma_\varepsilon\to 0$, we only can prove:

Theorem 6.1 *There exists $u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ such that*

$$u^\varepsilon \overset{\star}{\rightharpoonup} u \quad in \quad L^\infty(0,T;L^2(\Omega)) \tag{84}$$

$$u^\varepsilon \rightharpoonup u \quad in \quad L^2(0,T;H_0^1(\Omega)) \tag{85}$$

$$\frac{1}{|D_\varepsilon|}u^\varepsilon\chi_{D_\varepsilon}\to v_0 \quad in \quad \mathcal{D}'(\Omega) \quad a.e. \quad t\in[0,T] \tag{86}$$

where u is the only solution of the following problem:

$$\frac{\partial u}{\partial t}-\Delta u = f \quad in \quad \Omega^T \tag{87}$$

$$u(0) = u_0 \quad in \quad \Omega \tag{88}$$

Proof. The convergences (84)–(85) hold on some subsequences; they are insured by Proposition 2.3. We have to remark that (25) is the hypothesis which insures the existence of $v\in L^\infty(0,T;L^2(\Omega))$ which satisfies

$$\frac{1}{|D_\varepsilon|}u^\varepsilon\chi_{D_\varepsilon}\to v \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{a.e.} \quad t\in[0,T]$$

(see Lemma A-2 [3]); we have to prove that $v=v_0$.
Acting as usual, we take

$$\Phi^\varepsilon=(1-w_{R_\varepsilon})\varphi+w_{R_\varepsilon}G_{r_\varepsilon}(\psi) \tag{89}$$

for some $R_\varepsilon\in\mathcal{R}_\varepsilon$ and $\varphi,\psi\in\mathcal{D}(\Omega)$. Notice that in this case we have

$$\Phi^\varepsilon\to\varphi \quad \text{in} \quad H_0^1(\Omega) \tag{90}$$

because obviously $w_{R_\varepsilon}\to 0$ in $H_0^1(\Omega)$.
Passing to the limit in the variational formulation, we obtain in a straightforward manner

$$-\int_{\Omega^T}u\varphi\eta' dxdt-a\int_{\Omega^T}v\psi\eta' dxdt+\int_{\Omega^T}\nabla u\nabla\varphi\eta dxdt=\int_0^T\langle f,\varphi\rangle\eta dt+$$

$$+\left(\int_\Omega u_0\varphi dx + a\int_\Omega v_0\psi dx\right)\eta(0), \quad \forall \eta \in \mathcal{D}([0,T[)$$

Setting $\varphi = 0$, we find that v is independent of t and that $v \in C^0([0,T]; L^2(\Omega))$, which achieves $v = v_0$. Then, setting $\psi = 0$, we prove (87) and (88), the last one holding also in the sense of $C^0([0,T]; L^2(\Omega))$. ■

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